3420. [2009 : 109, 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\prod_{k=1}^{n}\left(\frac{(k+1)^2}{k(k+2)}\right)^{k+1} \;<\; n+1 \;<\; \prod_{k=1}^{n}\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1}.$$

Similar solutions by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam and Oliver Geupel, Brühl, NRW, Germany.

We recognize that the product on the left is a telescoping product, thus

$$\prod_{k=1}^{n}\left(\frac{(k+1)^2}{k(k+2)}\right)^{k+1} \;=\; \frac{2(n+1)^{n+2}}{(n+2)^{n+1}} \;=\; \frac{2(n+1)}{\left(1+\frac{1}{n+1}\right)^{n+1}}.$$

By the Bernoulli Inequality, we have

$$\left(1+\frac{1}{n+1}\right)^{n+1} \;>\; 1+(n+1)\frac{1}{n+1} \;=\; 2,$$

which proves the left inequality.

Again by the Bernoulli Inequality, we have

$$\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1} \;=\; \left(1+\frac{1}{k(k+1)}\right)^{k+1} \;>\; \left(1+\frac{1}{k}\right) \;=\; \frac{k+1}{k}.$$

Hence,

$$\prod_{k=1}^{n}\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1} \;>\; \prod_{k=1}^{n}\frac{k+1}{k} \;=\; n+1.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3421. [2009 : 109, 112] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $abc \le 1$. Prove that

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} \;\ge\; \frac{3}{2}.$$

Solution by Arkady Alt, San Jose, CA, USA.

After the substitution $(a,b,c) = \left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$, the inequality becomes equivalent to

$$\frac{y^2}{x(y+1)} + \frac{z^2}{y(z+1)} + \frac{x^2}{z(x+1)} \ \geq \ \frac{3}{2}\,,$$

where x, y, and z are positive real numbers with $xyz \geq 1$.

Since by the Cauchy-Schwartz Inequality we have

$$\big((xy+x)+(yz+y)+(zx+z)\big)\left(\frac{y^2}{x(y+1)} + \frac{z^2}{y(z+1)} + \frac{x^2}{z(x+1)}\right) \geq \ (x+y+z)^2\,,$$

it suffices to prove that

$$\frac{(x+y+z)^2}{xy+yz+zx+x+y+z} \ \geq \ \frac{3}{2}\,,$$

or

$$2(x+y+z)^2 \ \geq \ 3(xy+yz+zx)+3(x+y+z)\,.$$

However, the last inequality follows immediately from

$$(x+y+z)^2 \ \geq \ 3(xy+yz+zx)\,,$$

and

$$\begin{aligned}(x+y+z)^2 &= (x+y+z)(x+y+z) \\ &\geq 3\sqrt[3]{xyz}(x+y+z) \\ &\geq 3(x+y+z)\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect solution and two incomplete solutions submitted.

3422. [2009 : 110, 112] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c \leq 1$. Prove that

$$\frac{a}{a^3+a^2+1} \ + \ \frac{b}{b^3+b^2+1} \ + \ \frac{c}{c^3+c^2+1} \ \leq \ \frac{27}{31}\,.$$